Calculus 2

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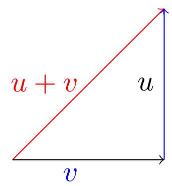
Vectors

Definition 1 ***Vector addition*** *If u and v are positioned so the initial point of v is at the terminal point of u, then the* ***sum*** *$u+v$ is the vector from the initial point of u to the terminal point of v.*



Definition 2 ***Scalar multiplication*** *If c is a scalar and v is a vector, then the scalar multiple cv is the vector whose lenght is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite id $c < 0$. If $c = 0$ or $v = 0$, then $cv = 0$*

If we place the initial point of a vector a at the origin of a rectangular coordinate system, then the terminal point of a has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two or three dimensional. These coordinates are called the **components** of a and we write

$$a = \langle a_1, a_2 \rangle \quad a = \langle a_1, a_2, a_3 \rangle$$

Definition 3 *Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, y_2)$, the vector a with representation $\vec{AB}$ is*

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

The **magnitude** or **length** of the vector v is the length of any of its representations and is denoted by the symbol $|v|$ or $\|v\|$.

Definition 4 *The length of the two dimensional vector $a = \langle a_1, a_2 \rangle$*

$$|a| = \sqrt{a_1^2 + a_2^2}$$

The length of the three dimensional vector $a = \langle a_1, a_2, a_3 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Definition 5 ***Unit vector*** *Unit vector is a vector whose length is 1 if the vector $a \neq 0$*

$$u = \frac{1}{|a|}a = \frac{a}{|a|}$$

Definition 6 ***Te standard basis vectors in*** V_3

$$i = \langle 1, 0, 0 \rangle \quad j = \langle 0, 1, 0 \rangle \quad k = \langle 0, 0, 1 \rangle$$

Definition 7 ***Dot Product*** *If* $a = \langle a_1, a_2, a_3 \rangle$ *and* $b = \langle b_1, b_2, b_3 \rangle$*, then the dot product of* a *and* b *is the number* $a \cdot b$ *given by*

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note: the dot product is also called the scalar product or **inner product**

Properties of the Dot Product if a, b, c are vectors in V_3 and d is a scalar, then

1. $a \cdot a = |a|^2$
2. $a \cdot (b + c) = a \cdot b + a \cdot c$
3. $0 \cdot a = 0$
4. $a \cdot b = b \cdot a$
5. $(ca) \cdot b = c(a \cdot b) = a \cdot (cb)$

Theorem 1 *If* θ *is the angle between the vectors* a *and* b*, then*

$$a \cdot b = |a| \cdot |b| \cos \delta$$

Corollary 1

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

Two nonzero vectors a and b are called **perperdicular** or **orthogonal** if the angle between them is $\theta = \pi/2$

Definition 8 ***Orthogonal*** *Two vectors* a *and* b *are orthogonal if and only if* $a \cdot b = 0$

Definition 9 ***Direction angles*** *The direction angles of nonzero vector* a *are the angles, in the interval [0,π], that* a *makes with the positive* x, y, z *axes.*

Definition 10 ***Direction cosine*** *The cosine of the direction angles are called the direction cosines of the vector* a

Note: The direction cosines of a are the components of the unit vector in direction of a

$$\frac{1}{|a|} a = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Definition 11 ***Scalar projection***
Scalar projection of b *onto* a *(or the component of* b *along* a*):*

$$comp_a b = \frac{a \cdot b}{|a|}$$

Note: $|b|\cos\theta = \text{comp}_a b$

Definition 12 ***Vector projection***
Vector projection of b onto a:

$$proj_a b = \left(\frac{a\cdot b}{|a|}\right)\frac{a}{|a|} = \frac{a\cdot b}{|a|^2}a$$

Definition 13 ***Distance from a point to a line*** *Let $P_1(x_1,y_1)$ and $ax+by+c=0$ be the line. Then, the distance between them is given by*

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Definition 14 ***Cauchy-Schwartz Inequality*** *Let a and b be two vectors, then*

$$|a\cdot b| \le |a||b|$$

Definition 15 ***Triangle inequality*** *Let a and b be two vectors. Then*

$$|a+b| \le |a|+|b|$$

Definition 16 ***Parallelogram identity*** *Let a and b be two vectors, then*

$$|a+b|^2+|a-b|^2 = 2|a|^2+2|b|^2$$

Definition 17 ***Cross Product*** *If $a=\langle a_1,a_2,a_3\rangle$ and $b=\langle b_1,b_2,b_3\rangle$, then the cross product of a and b is the vector*

$$a\times b = \langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$$

An easy way to remember is to use determinants, by rewrite the definition using second order determinants and the standard basis vectors i, j, k:

$$\begin{aligned} a\times b &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k \\ &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

Theorem 2 *The vector $a\times b$ is orthogonal to both a and b*

Theorem 3 *If θ is the angle between a and b, then the length of the cross product $a\times b$ is given by*

$$|a\times b| = |a||b|\sin\theta$$

Corollary 2 *Two non zero vectors a and b are parallel if and only if*

$$a\times b = 0$$

Note: the length pf the cross product $a\times b$ is equal to the area of the parallelogram determined by a and b

Theorem 4 *The volume of the parallelepiped determined by the vectors a, b, c is the magnitude of their scalar triple product:*

$$V = |a\cdot(b\times c)| = \left|\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|$$

Equations of Lines and Planes

Definition 18 ***Line*** *A line L in three dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and a direction for L, which is conveniently described by a vector v parallel to the line.*

Definition 19 ***Vector equation of a line*** *Let $r_0 = \langle x_0, y_0, z_0 \rangle$ and $r = \langle x, y, z \rangle$ be the position vctors of two points P_0, P in a line L. Then, if vector a with representation $\vec{P_oP}$ gives*

$$r = r_0 + a$$

Since a and v are parallel vectors, there exists a scalar t s.t $a = tv$

$$r = r_0 + tv$$

Definition 20 ***Parametric equations*** *The parametric equations for a line through a point (x_0, y_0, z_0) and parallel to the direction vector $v = \langle a, b, c \rangle$ are*

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Definition 21 ***Line segment*** *If r_0 and r_1 are two different vectors on a line L, then we can take $v = r_1 - r_0$ and the vector equation of L becomes*

$$r(t) = (1-t)r_0 + tr_1 \qquad 0 \leq t \leq 1$$

This is called the line segment from r_0 to r_1.

Planes

Definition 22 *A **plane** in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector n (normla vector) that is orthogonal to the plane.*

Definition 23 ***Scalar equation of the plane*** *A scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$ is*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Definition 24 ***The distance between a point and a plane*** *The distance between a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cylinders and Quadric Surfaces

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** of the surface.

Definition 25 ***Cylinder*** *A cylinder is a surface that consist of all lines (called **rulings**) that are parallel to a given line pass through a given plane curve*

Definition 26 ***Quadric surface*** *A quadric surface is the graph of a second degree equation in three variables x, y and z. The most general equation is*

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Where $A, B, C, ..., J$ are constants, but by traslation and rotation it can be brought into one of the two stardard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad or \quad Ax^2 + By^2 + Iz = 0$$

Note: Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

Vectors Functions

In general, a function is a rule that assigns to each element in the domain an element in the range. A **Vector-valued function**, or **vector function**, is simply a function whose domain is set of real numbers and whose range is a set of vecotrs.

It means that for every number t, the independent variable, in the domain of r there is a unique vector in V_3 (if we are in three dimension) denoted by $r(t)$. If $f(t), g(t), h(t)$ are the components of the vectors $r(t)$, then f, g, h are real valued functions called the component functions of r and we write

$$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$$

Definition 27 *$L \in \mathbb{R}^n$ is called limit of $r : I \in \mathbb{R} \to \mathbb{R}^n$ at $a \in \mathbb{R}$ if*

$$\forall \epsilon > 0 \, \exists \delta > 0 \quad s.t. \quad 0 < |t - a| < \delta \Rightarrow \|r(t) - L\| < \epsilon$$

Notation:

$$\lim_{t \to a} f(t) = L$$

Lemma 1 *let $r(t) = \langle f(t), g(t), h(t) \rangle$, then*

$$\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Provided the limits of the component functions exist

Definition 28 ***Continuity*** *$r : I \to \mathbb{R}^n$ is continuous at $a \in I$ if*

$$\lim_{t \to a} r(t) = r(a)$$

Lemma 2 *r is continuous at $t = a$ if the components functions are continuous at a*

Definition 29 ***Space Curves*** *Let $I \subset \mathbb{R}$ interval, $r : I \to \mathbb{R}^3$ continuous. Then $C : r(I)$ is called space curve.*

r is called ***parametrization*** *of C*

Definition 30 *$r : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ interval, is called **differentiable** at $t \in I$ if the limit*

$$r'(t) = \frac{dr}{dt}(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

exists. Thus $r'(t)$ is called derivative of r at t

Remark: derivative of $r : I \to \mathbb{R}^n$ at $t \in I$ is the tangent vector of the curve $C' = r(I)$ at the point $r(t)$

And if $r'(t) \neq 0$ we can define the unit tangent vector

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Moreover, note that $r(t+h) - r(t)/h$ gives the average velocity over time interval of length h and its limit is the **velocity vector** $v(t)$ at time t.

The velocity is also the tangent vector and points in the direction of the tangent line. The speed of the particle at time t is the magnitude of the velocity vector $\|v(t)\| = \|r'(t)\|$. The acceleration, instead, is the derivative of the velocity.

Theorem 5 *$r : I \to \mathbb{R}^n$, $r \langle f, g, h, ...\rangle$ is differentiable iff each components are differentiable*

Definition 31 *$r : I \to \mathbb{R}^3$ is integrable $\Leftrightarrow$ f, g, h are integrable and for $a, b \in I$*

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt\right) i + \left(\int_a^b g(t)dt\right) j + \left(\int_a^b h(t)dt\right) k$$

Note: It is used to determinate a curve as **class** C^n if such curve is differentiable and his n-th derivative is continuous.

Arc Length

Let $I = [a, b]$, $r : I \to \mathbb{R}^3$ continuous and differentiable, i.e $r'(t)$ exists and continuous.

Definition 32 *The length of $C' = r(I)$ with $C' -$ parametrization. $r : [a, b] \to \mathbb{R}^3$ is define with*

$$L = \int_a^b \|r'(t)\| dt$$

Definition 33 *let $r : [a, b] \to \mathbb{R}^n$, such that r' is continuous*

$$s(t) = \int_a^t \|r'(u)\| du \rightsquigarrow t(s)$$

In other words, $s(t)$ is the length of the part of C between $r(a)$ and $r(t)$.

Note: it is often useful parametrize a curve with respect to arch length because the arch length does not depend of the coordinate system or a particular parametrization. Then, the **parametrization by arch length** is given by

$$r(s) = r(t(s))$$

Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of s along the curve. Note as well that we will start the measurement of distance from where we are at $t = a$

Curvature

Definition 34 *A curve C is* ***smooth*** *if it has a C' parametrization $r : I \to \mathbb{R}^n$ with $r'(t) \neq 0$ for all $t \in I$*

Definition 35 *The* ***curvature*** *if a smooth curve C with a C^2 parametrization $r : I \to \mathbb{R}^n$ is*

$$K = \left\| \frac{dT}{ds} \right\|$$

where T is the unit tangent vector

Note: By the chain rule the curvature is given by

$$K = \frac{\|T'(t)\|}{\|r'(t)\|}$$

Moving frame and torsion

Let C be a smooth curve with C^3 paramets. $r : I \to \mathbb{R}^3$
Goal: define in some natural way three mutually orthogonal vectors of length 1 at each point on C

The first vector: the unit tangent vector T

$$T(s) = \frac{dr}{ds} = \frac{r'(t)}{\|r'(t)\|}$$

The second vector: The <u>principal unit normal vector</u> (or simply unit normal) $N(t)$:

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Note: $\frac{dT}{ds} = KN$. Moreover, $T(t)$ and $N(t)$ define a plane, called the osculating plane (from latin osculum which means kiss). The circle of curvature, or the osculating circle, of a curve C at point P is the circle in the osculating plane that passes through P with radius $1/K$ and center a distance $1/K$ from P along the vector N. We can think about it as the circle that best describes how C behaves near P

The third vector: The <u>binormal vector</u> $B(t)$

$$B(t) = T(t) \times N(t)$$

Then (T, N, B) is a right handed system of orthogonal unit vectors (frame)

Definition 36 ***Torsion*** *The torsion(τ) measure how spatial (non-planer) a curve is*

$$\frac{dB}{ds} = -\tau N$$
$$\tau = -\frac{dB}{ds} \cdot N$$

Definition 37 ***Frenet-Serret Formulas***

$$1.\quad \frac{dT}{ds} = KN$$
$$2.\quad \frac{dN}{ds} = -KT + \tau B$$
$$3.\quad \frac{dB}{ds} = -\tau N$$

In matrix notation:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Remark: $r : I \subset \mathbb{R} \to \mathbb{R}^n$ is differentiable at $t \in I$

$$\Leftrightarrow \exists v \in \mathbb{R}^n \text{ such that } \lim_{h\to 0} \frac{r(t+h) - r(t)}{h} = v$$
$$\Leftrightarrow \exists v \in \mathbb{R}^n \text{ such that } \lim_{\tau\to t} \frac{r(\tau) - r(t)}{\tau - t} = v$$
$$\Leftrightarrow \exists v \in \mathbb{R}^n \text{ such that } \lim_{\tau\to t} \frac{\|r(\tau) - (r(t) + v(\tau - t))\|}{|\tau - t|} = 0$$

Note: $t \to r(t) + v(\tau - t)$ is the linear approximation of r at $r(t)$

Partial derivatives

Functions of two variables

Definition 38 ***Function of two variables*** *A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D\}$*

Note: we often write $z = f(x, y)$, where x, y are the independent variables and z the dependent one. Moreover, this kind of function have as domain a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$.

Definition 39 ***Graph*** *if f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D*

Definition 40 *The* **level curves** *of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant (in the range of f).*

A collection of level curves is called ***contour map***

Limits and Continuity

Definition 41 ***Limit*** *Let f be a function of two variables whose domain D includes points arbitrary close to (a, b). Then we say that the* ***limit of*** *$f(x, y)$* ***as*** *(x, y)* ***approaches*** *(a, b) is L and we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

If for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{and} \quad |f(x, y) - L| < \epsilon$$

Here follow the step for show that a limit does not exists:

1. if $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1
2. if $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2
3. if $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exists

Definition 42 ***Continuity*** *A function f is continuous at (a, b) if*

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is continuous on D if f is continuous at every point (a, b) in D

Definition 43 ***Limit of function with two or three variables*** *if f is defined on $D \subset \mathbb{R}^n$, then $\lim_{x\to a} f(x) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that*

$$\text{if } x \in D \quad \text{and} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Partial derivatives

Definition 44 ***Parial derivatives*** *If f is a function of two variables, its* ***partial derivatives*** *are the function f_x and f_y defined by*

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notation:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2$$

Theorem 6 ***Higher derivatives*** *$f : D \in \mathbb{R}^n \to \mathbb{R}$ is called C^k if all partial derivatives at to order k exists and are continuous.*

Theorem 7 ***Clairaut's theorem*** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Note: it means that the order of the partial derivative of class C^2 it does not matter.

Tangent planes

Definition 45 ***Tagent plane*** *Suppose $f : D \in \mathbb{R}^2 \to \mathbb{R}$ and is it of class C^2, then an equation of the plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear approximation

Definition 46 ***Linearization and linear approximation*** *The linear function whose graph is the tanget plane, namely*

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

*is called **linearization** of f at (a, b) and the approximation*

$$L(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

*is called the **linear approximation** or the tangent plane approximation of f at (a, b)*

Theorem 8 *f is called **differentiable** at a if the partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exists for all $i = 1, ..., n$ and for*

$$L(x) = f(a) + f_{x_i}(a)(x_i + a_i) + \cdots + f_{x_n}(a)(x_n + a_n)$$

if the following limits exists

$$\lim_{x \to a} \frac{f(x) - L(x)}{\|x - a\|} = 0$$

Definition 47 ***Gradient*** *The derivative of f at a is given by $(f_{x_i}(a), ..., f_{x_n}(a))$ and is denoted as **gradient** of f at a: $\nabla f(a)$*

Theorem 9 *$f : D \subset \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives at $a \in D$. Then f is differentiable at a*

Remark:

- a function is differentiable if it can be well approximated by its linearization
- the graph of the linearization L is the tangent plane.
- the derivative of f at a is a linear map $\mathbb{R}^n \to \mathbb{R}$, $\Delta x \to \nabla f(a)\Delta x$

Theorem 10 *$f : D \subset \mathbb{R}^n \to \mathbb{R}$ differentiable at $a \in D$. Then f is continuous at a*

Remark: the existence of partial derivatives is a rather weak property. It does not imply differentiability nor even continuity

The Chain Rule

Recall that the Chain Rule for functions for a single variable gives the rule for differating a composite function: if $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

Theorem 11 ***Chain Rule case 1*** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differetiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Theorem 12 ***Chain Rule general*** *$r(t) = (x_1(t), ..., x_n(t))$ where $t \in I$ and $f : D \subset \mathbb{R}^n \to \mathbb{R}$, thus $r(I) \subset D$. Then*

$$\begin{aligned}\frac{d}{dt}(f \circ r)(t) &= \frac{d}{dt} f(x_1(t), ..., x_n(t)) \\ &= \frac{\partial f}{\partial x_1}(r(t))\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}(r(t))\frac{dx_n}{dt} \\ &= \nabla f(r(t))r'(t)\end{aligned}$$

Even more generally:

Theorem 13 *Let $g : \mathbb{R}^m \to \mathbb{R}^n$, where $t = (t_1, ..., t_m) \mapsto g(t) = (x_1(t), ..., x_n(t))$ and let $f : \mathbb{R}^n \to \mathbb{R}$, then $z = (f \circ g)$. Moreover, suppose that z is a differentiable function of n variables and each variables are differentiable function of the m variables. Then,*

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

where $i = 1, ..., m$

Implicit Differentiation

Assume that $f(x, y) = 0$ can be locally be showed for y, i.e. $y = y(x)$ s.t $f(x, y(x)) = 0$ f.a $x \in I$. Now let define $f(x(t), t(t)$ with $x(t) = t)$ and $r(t) = (x(t), y(t))$. Then $(f \circ r)(t) = 0$ f.a. t. If f is differentiable, we can apply the Chain Rule, we obtain:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dt = 1$, so if $f_y \neq 0$ we can solve for y' and obtain:

$$y' = -\frac{f_x}{f_y}$$

Now, for generalize it suppose we have $f(x,y,z) = 0$ that can be locally solved for z, i.e. $z = (x,y)$. Then, $f(x,y,z(x,y)) = 0$ f.a (x,y). Then by apply the chain rule we obtain:

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $\partial/\partial x(x) = 1$ and $\partial/\partial y(y) = 0$, so the equation becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $f_z \neq 0$, we solve $\partial z/\partial x$ and obtain

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

The formula for $\partial z/\partial y$ is obtain in a similar manner

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

Theorem 14 ***Implicit Function Theorem*** *Let $F : D \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let $a = (a_1, ..., a_n)$ be contained in the level set*

$$S := \{x \in D | F(x) = c\}$$

for some $c \in \mathbb{R}$. If $F_{x_n}(a) \neq 0$, then there exists a neighborhood U of $(a_1, ..., a_{n-1})$ in $\mathbb{R}^{n-1}$ and a neighborhood V of a_n in $\mathbb{R}$ and a function $f : U \to V$ of class C^1 such that if $(x_1, ..., x_{n-1}) \in U$ and $x_n \in V$ satisfy $F(x_1, ..., x_{n-1}, x_n) = c$ then $x_n = f(x_1, ..., x_{n-1})$. The function f is then called implicit function. It holds that

$$\frac{\partial f}{\partial x_k} = -\frac{\frac{\partial F}{\partial x_k}}{\frac{\partial F}{x_n}}$$

for $k = 1, ..., n-1$.

Direction derivatives

Definition 48 *Let $u = (u_1, ..., u_n) \in \mathbb{R}^n$ be unit vectors and $f : D \subset \mathbb{R}^n \to \mathbb{R}$ with $x = (x_1, ..., x_n) \mapsto f(x)$ then*

$$D_u f(x) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

direction derivatives of f in the direction of u at x

Theorem 15 *Suppose f is differentiable then*

$$D_u f(x) = \frac{\partial f}{\partial x_1}u_1 + \cdots + \frac{\partial f}{x_n}u_n = \nabla f(x) u$$

Remarks:

- $D_u f(x)$ is the increase of f in the direction of u

- The existence of the direction derivatives in all direction u does not imply differentiability

Theorem 16 *$D_u f(x)$ is maximal for*

$$u = \frac{1}{\|\nabla f(x)\|}\nabla f(x)$$

Note: $D_u f(x)$ is maximal when is equal $\|\nabla f(x)\|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x)$

Theorem 17 *$f : D \subset \mathbb{R}^n \to \mathbb{R}$ differentiable, $c \in \mathbb{R}$ and*

$$S = \{(x_1, ..., x_n) \in D | f(x_1, ..., x_n) = c\}$$

Let $a \in S$. Then $\nabla f(a)$ is perperdicular to S at a

Remark: The gradient vector at a point P, $\nabla f(x_0, y_0, z_0)$, is perpendicular to the tangent vector $r'(t_0)$ to any curve C on S that passes through P.

Maximum and Minimum values

Definition 49 *A point $a \in D$ of a C^1 function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is called **critical point** if $\nabla f(a) = 0$*

Theorem 18 *Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 and has a critical point $(a, b) \in D$. Let $d = det\, Hessf(a, b)$ where*

$$Hessf(a, b) = \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}$$

*Which is called the **Hessian matrix** of f at (a, b), then the determinent $d = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b)^2$:*

- *If $d > 0$ and $f_{xx}(a, b) > 0$, then f has local minimum at (a, b)*
- *if $d > 0$ and $f_{xx}(a, b) < 0$, then f has local maximum at (a, b)*
- *if $d < 0$ then f has a saddle at (a, b) where a saddle is a critical point and at which each negb. contains points (x, y) with $f(x, y) < f(a, b)$ and points (x, y) with $f(x, y) > f(a, b)$*

Theorem 19 ***Extreme Value Theorem*** *If f is continuous on a closed, bounded set $D \in \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D*

Theorem 20 ***Lagrange multiplier*** *Let $D \subset \mathbb{R}^n$ and $f, g : D \to \mathbb{R}$ of class C^1 and let $S = \{x \in D | g(x) = c\}$. Then if $f_{|S}$ (restriction of f to S) has an extrimum at $a \in S$ where $\nabla g(a) \neq 0$, then*

$$\exists \lambda \in \mathbb{R} \quad s.t \quad \nabla f(a) = \lambda \nabla g(a)$$

Theorem 21 *$f, g, h : D \subset \mathbb{R}^n \to \mathbb{R}$ of class C^1. Let $S = \{x \in D | g(x) = c$ and $h(x) = d\}$. Then if $f_{|S}$ has an extrimum at $a \in S$ then*

$$\exists \lambda, \mu \in \mathbb{R} \quad s.t \quad \nabla f(a) = \lambda \nabla g(a) + \mu \nabla h(a)$$

Multiple Integration

Double integral

Definition 50 ***Double Integral*** *The double integral of f over the rectangle R is*

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

If this limit exists

Note: the double integral is the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$

Theorem 22 ***Fubini's theorem*** *if f is continuous on the rectangle*

$$R = \{ (x,y) \,|\, a \le x \le b,\ c \le y \le d \}$$

Then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exists

General Regions

Consider a general region D. Suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R. In order to integrate a function f over D we define a new function F with domain R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

Then if F is integrable over R, then we define the double integral of f over D by

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

There exists three type of plane region D:

- A plane region D is said to be of **Type 1** if it lies between the graphs of two continuous functions of x, that is

  $$D = \{ (x,y) \,|\, a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$$

  where g_1 and g_2 are continuous on $[a,b]$

- A plane region D is said to be of **Type 2** if it lies between the graphs of two continuous functions of y, that is

  $$D = \{ (x,y) \,|\, c \le y \le d,\ h_1(y) \le x \le h_2(y) \}$$

  where g_1 and g_2 are continuous on $[c,d]$

- A plane region D is said to be of **Type 3** if it is both of type 1 and type 2

Remarks: For a type 1 region, the functions must be continuous but they do not need to be defined by a single formula

Definition 51 ***Type 1*** *if f is continuous on a type 1 region D then,*

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Definition 52 ***Type 2*** *if f is continuous on a type 2 region D then,*

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Polar Coordinates

The polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) of a point by the equations

$$r^2 = x^2 + y^2 \quad x = r\cos(\theta) \quad y = r\sin(\theta)$$

Definition 53 *If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta$$

Triple integrals

Just as we defined single integrals for functions of one variables and double integrals for functions of two variables, so we can define triple integrals for functions of three variables

Definition 54 ***Triple Integral*** *The double integral of f over the box B is*

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

If this limit exists

Note: The triple integrals can represent volume of 3 dimensional object. But we have already the double integral for the volume. So, why the triple integral?
First note that if $f(x,y,z) = 1$ then the triple integral in a given surface is indeed the volume of such surface. Then, it easier then the double because we do not have a function.
The most important feature of the triple integral is that its represent the **total mass** of the given surface. Indeed, the function $f(x,y,z)$ represent the density at each midpoint of the little boxes.

Theorem 23 ***Fubini's theorem*** *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Remark: Like for the double integral, also the triple integral can be solved in an arbitrary order.

General Regions

Now we define the triple integral over a general bounded region E in three-dimensional space (solid) as we did for the double integral. Therefore, we enclose E in a box B such that F agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

As for the double integral, there exist four types:

1. **Type 1** if it lies between graphs of two continuous functions of x, y

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dA$$

2. **Type 2** if it lies between graphs of two continuous functions of z, y

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\, dA$$

3. **type 3** if the projection lies on the xz-plane

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\, dA$$

4. **Type 4** if its all of them

Cylindrical Coordinates

In the cylindrical coordinates system, a point P in the three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the $xy-$plane and z is the directed distances from the $xy-$plane to P.
The formula are the same as the polar coordinates, the only difference is the $z = z$. Then the triple integral is given by

$$\iiint_D f(x, y, z)\, dV = \iiint f(r\cos(\theta), r\sin(\theta), z) r\, dr\, d\theta\, dz$$

Spherical Coordinates

The spherical coordinates of a point P are: (ρ, θ, ϕ), where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

The spherical coordinates are useful in problems where there is symmetry about a point, and the origin is placed at this point.

Here follow the conversion equations

$$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho \sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

Consequently, we have arrived at the formula for the triple integration in spherical coordinates

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi)\,|\,a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

Change of Variables

In one dimensional Calculus we define the following Substitution Rule

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$

Definition 55 ***Jacobian*** *The Jacobian of the transformation T given byb $x = g(u,v)$ and $y = h(u,v)$ is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can give an approximation of the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

Theorem 24 *Suppose that T is a C^1 transformation whose jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R. Suppose that T is one to one. Then*

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

For the triple integral is close the same, where the Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

It follows that the previous theorem will be

Theorem 25

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Vectors Calculus

Vector Fields

Definition 56 ***Vector Field*** *Let $D \subset \mathbb{R}^n$, a vector field on $\mathbb{R}^n$ is a function $\boldsymbol{F}$ that assigns to each point (x,y) in D a n-dimensional vector $\boldsymbol{F}(x_1,...,x_n)$*

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x,y)$ starting at the point (x,y). Note that $\mathbf{F}$ can be written in the following way (i.e in n=2):

$$\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i} + Q\mathbf{j}$$

Definition 57 *A vector field $\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is called* ***gradient vector field*** *or* ***conservative*** *if there exists a function $f : D \to \mathbb{R}$ with $\boldsymbol{F} = \nabla f$. In this case f is called* ***potential function*** *for $\boldsymbol{F}$*

Line integrals

Let C be a smooth curve in $\mathbb{R}^n$ with parameters $r : [a,b] \to \mathbb{R}^n$ and $t \mapsto r(t)$ with $r'(t) \neq 0$ f.a $t \in [a,b]$. Then length of C is

$$L = \int_a^b \|r'(t)\|\,dt = \int_0^L ds$$

where s is the archlength

Definition 58 ***Line integral*** *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ continuous. Then the line integral of f along $C \subset D$ is defined as*

$$\int_C f\,ds = \int_a^b f(r(t))\|r'(t)\|\,dt$$

Definition 59 ***Piecewise-smooth curve*** *C is called piecewise smooth if C is a union of finitely many smooth curves C_i for $i = 1,...,n$ with the initial point of C_{i+1} is the terminal point of C_i.*
Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \sum_{i=1}^n \int_{C_i} f(x,y)\,ds$$

Theorem 26 *Line integrals of scalar valued functions are independent of parametrization (and in particular independent of the orientation[1])*

[1] direction of the arrow on the curve

Definition 60 *Let $\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vectors fields and C be a smooth curve, $C \in D$, with parametrization $r : [a,b] \to \mathbb{R}^n$. Then the line integral of $\boldsymbol{F}$ along C is defined as*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(r(t)) \cdot r'(t)\, dt = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds$$

Note: Assume $\mathbf{F} \in \mathbb{R}^3$ is given by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Moreover, note that an interpretation of line integral is the required work for move a particle along C

Theorem 27 *$\boldsymbol{F}$ continuous vectors field on $D \subset \mathbb{R}^n$ where D contains a smooth curve C. Let $r : [a,b] \to \mathbb{R}^n$ and $h : [c,d] \to \mathbb{R}^n$ be parametrizations of C. Then*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \begin{cases} \int_C \boldsymbol{F} \cdot d\boldsymbol{h} & \text{If } r(a) = h(c) \text{ and } r(b) = h(d) \\ -\int_C \boldsymbol{F} \cdot d\boldsymbol{h} & \text{If } r(a) = h(d) \text{ and } r(b) = h(c) \end{cases}$$

Note: the integral is the same if r and h gives the same orientiation, otherwise is the opposite.

Theorem 28 *Let C be a smooth curve given by the vector function $r(t)$, $a \le t \le b$. Let f be a differentiable function such that $\boldsymbol{F} = \nabla f$. Suppose $\boldsymbol{F}$ is continuous. Then*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(r(b)) - f(r(a))$$

Definition 61 *$\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ continuous vector field is* ***independent of path*** *if*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_H \boldsymbol{F} \cdot d\boldsymbol{h}$$

for all smooth curves C and H with the same initial and final points

Theorem 29 *Line integrals of conservative vector fields are independent of path*

Theorem 30 *$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is* ***independent of path*** *in D if and only if $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for every closed path C in D*

Definition 62 ***Connected*** *$D \subset \mathbb{R}^n$ is called* ***connected*** *if any two points in D can be connected by a curve that is contained in D*

Theorem 31 *Let $D \subset \mathbb{R}^n$ connected, $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ vector field. If $\boldsymbol{F}$ independent of path then $\boldsymbol{F}$ is conservative, i.e there is $f : D \to \mathbb{R}$ with $\boldsymbol{F} = \nabla f$*

Remark: $\mathbf{F}$ is conservative on $D \subset \mathbb{R}^2$, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = f_x\mathbf{i} + f_y\mathbf{j}$ where f is the potential function

Definition 63 *$D \subset \mathbb{R}^n$ is called* ***simply connected*** *if it is connected and all closed curves in D can be contracted to a point in D without leaving D*

Theorem 32 *If $\boldsymbol{F}$ continuous vector field and $\boldsymbol{F}$ independent of path, then is equivently to say that*

$$\oint_C \boldsymbol{F} \cdot dr = 0$$

for all closed curves C

Note: C is closed if initial and final point are the same.

Theorem 33 *Let $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ be a vector field on a symply connected domain D in $\mathbb{R}^2$ with P and Q being C^1. Then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Leftrightarrow \boldsymbol{F} \text{ is conservative}$$

Green's Theorem

A planas curve C is called **simple** if it has no selfintersection between the end points.

Theorem 34 *Let D be a bounded domain in $\mathbb{R}^2$ whose boundaries ∂D consist of finitely many, simple, closed, picewise C^1 curves. If P and Q have continuous partial derivatives on a open region that contains D, then*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Diverge and Curl

Definition 64 *The **Del operator**, denoted ∇, is a vector differential operator*

Note: $\mathbb{R}^3$ it is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Definition 65 *The **divergence** of $\boldsymbol{F}$, denote $\nabla \cdot \boldsymbol{F}$, is defined by*

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Note: The value of the diverges can be interpreted as the rate of the net mass flow of $\mathbf{F}$ in if its negative, out if its positive

Definition 66 *the **curl** of $\boldsymbol{F}$, denoted $\nabla \times \boldsymbol{F}$ is defined by*

$$\begin{aligned}\nabla \times \boldsymbol{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (P, Q, R) \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k}\end{aligned}$$

Remark: The value of the curl measures the local rate of rotation of $\mathbf{F}$ at x

- The **Direction** of the curl is the orientation of the local rotation of $\mathbf{F}$ at x

- the **Magnitude** of the curl is the rate of this local rotation

Theorem 35 *If f is a scalar valued function of class C^2, then*

$$\nabla \times (\nabla f) = 0$$

Theorem 36 *If $\boldsymbol{F}$ is a vector field of class C^2 on $D \subset \mathbb{R}^3$, then*

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$$

Theorem 37 *If $\boldsymbol{F}$ is a vector field in $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\nabla \times \boldsymbol{F} = 0$, then $\boldsymbol{F}$ is a conservative vector field*

Theorem 38 *For a C^1 vector field $\boldsymbol{F}$ on $D \subset \mathbb{R}^3$ where D is simply connected, it holds that $\boldsymbol{F}$ is conservative $\Leftrightarrow$ $curl\boldsymbol{F} = 0$*

Note: We can rewrite the equation in the Green Theorem in the vector form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (curl\mathbf{F}) \cdot \mathbf{k}\, dA$$

Which express the line integral of the tangential component of $\mathbf{F}$ along C as the double integral of the vertical component of $curl\mathbf{F}$ over the region D enclose by C

Parametrized Surfaces in $\mathbb{R}^3$

Definition 67 *A **parametrized surface** in $\mathbb{R}^3$ is a continuous map $X : D \to \mathbb{R}^3$ that is one-one on D, except possibly along ∂D*

Definition 68 *If $D \subset \mathbb{R}^2$ is a connected open set, possibly toghether with some of its boundary points, then the graph of a continuous, scalr-valued function $f : D \to \mathbb{R}$ may be parametrized by $X : D \to \mathbb{R}^3$; $X(s,t) = (s, t, f(s,t))$*

Definition 69 *If $X(s,t) = (x(s,t), y(s,t), z(s,t))$ is differentiable at $s_0, t_0 \in D$, then a **tangent vector** $T_s(s_0,t_0)$ to the s-coordinate curve $X(s_0,t)$ at (s_0,t_0) is:*

$$T_s(s_0,t_0) = \frac{\partial X}{\partial s}(s_0,t_0) = \frac{\partial x}{\partial s}(s_0,t_0)\boldsymbol{i} + \frac{\partial y}{\partial s}(s_0,t_0)\boldsymbol{j} + \frac{\partial z}{\partial s}(s_0,t_0)\boldsymbol{k}$$

Similarly, a tangent vector $T_t(s_0,t_0)$ to the t-coordinate curve $X(s,t_0)$ at (s_0,t_0) is given by

$$T_t(s_0,t_0) = \frac{\partial X}{\partial t}(s_0,t_0) = \frac{\partial x}{\partial t}(s_0,t_0)\boldsymbol{i} + \frac{\partial y}{\partial t}(s_0,t_0)\boldsymbol{j} + \frac{\partial z}{\partial t}(s_0,t_0)\boldsymbol{k}$$

Definition 70 *$S = X(D)$ is **smooth** at $X(s_0,t_0)$ if X is of class C^1 and $T_s(s_0,t_0) \times T_t(s_0,t_0) \neq 0$*

Definition 71 *If S is smooth at every point $X(s_0,t_0)$, then the nonzero vector*

$$N(s_0,t_0) = T_s(s_0,t_0) \times T_t(s_0,t_0)$$

*Is called the **standard normal vector***

Definition 72 *A* ***piecewise smooth*** *parametrized surface is the union of images of finitely many parametrized surfaces*

Definition 73 *Suppose $S = X(D)$ is a smooth parametrized surface, then the* ***surface area*** *of S is given by*

$$\iint_D \|T_s \times T_t\| \, ds \, dt = \iint_D \|N(s,t)\| \, ds \, dt$$

Note: $N(s,t) = T_s \times T_t = \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}$

Definition 74 *If S is the graph of a class C^1 function $f(x,y)$ (i.e, $S = X(D)$ where $X(s,t) = (s,t,f(s,t))$, $(s,t) \in D$). Then the surface area is given by*

$$\iint_D \sqrt{\left(\frac{\partial f}{\partial s}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2 + 1} \, ds \, dt$$

Theorem 39 *Let $X : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region; and let $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, with X containing $S = X(D)$. Then the* ***scalar surface integral*** *of f along X is*

$$\begin{aligned} \iint_X f \, dS &= \iint_D f(X(s,t)) \|T_s \times T_t\| \, ds \, dt \\ &= \iint_D f(X(s,t)) \|N(s,t)\| \, ds \, dt \end{aligned}$$

Theorem 40 *Let $X : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region and let $F : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field, with X containing $S = X(D)$ then the* ***vector surface integral*** *of F along X is given by*

$$\iint_X \boldsymbol{F} \cdot dS = \iint_D \boldsymbol{F}(X(s,t)) \cdot N(s,t) \, ds \, dt$$

Definition 75 *If $X : D \to \mathbb{R}^3$ is a smooth parametrized surface then we can define*

$$n(s,t) = \frac{N(s,t)}{\|N(s,t)\|}$$

called the ***unit normal vector*** *to $S = X(D)$*

Note:

$$\iint_X \mathbf{F} \cdot dS = \iint_X (\mathbf{F} \cdot n) \, dS$$

Significance: The vector surface integral represents the **flux** of $\mathbf{F}$ across $S = X(D)$. That is, if $\mathbf{F}$ represents the velocity field of a fluid in $\mathbb{R}^3$, the flux is the rate of fluid transported across S per unit time

Theorem 41 *Scalar surface integrals do not depend on the parametrization*

Definition 76 *A smooth, connected surface S is* ***orientable*** *if a single unit normal vector can be defined at each point so that the collection of these vectors varies continuously over S If S is orientable, then it has two orientations*

Theorem 42 *Vector surface integrals depend only on whether the reparametization is orientation-preserving. That is, if Y is a smooth reparametrization of X then*

$$\iint_Y \boldsymbol{F} \cdot dS = \begin{cases} \iint_X \boldsymbol{F} \cdot dS & \text{if } Y \text{ is orientation-preserving} \\ -\iint_X \boldsymbol{F} \cdot dS & \text{if } Y \text{ is orientation-reversing} \end{cases}$$

Theorem 43 ***Stokes's Theorem*** *Suppose that S is a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$, oriented by unit normal n at each point; and that ∂S consists of finitely many picewise C^1 simple closed curves, and that $\boldsymbol{F}$ is a vector field of class C^1 whose domain includes S, then*

$$\iint_S \nabla \times \boldsymbol{F} \cdot dS = \oint_{\partial S} \boldsymbol{F} \cdot ds$$

Theorem 44 ***Gauss' Divergence theorem*** *Suppose:*

- *D is a bounded solid region $\mathbb{R}^3$*
- *∂D consists of finitely many picewise smooth, closed, orientable surfaces, each oriented by unit normals that point away from D*
- *$\boldsymbol{F}$ is a vector field of class C^1 whose domain includes D*

then

$$\iiint_D \nabla \cdot \boldsymbol{F} dV = \iint_{\partial D} \boldsymbol{F} \cdot dS$$